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Module Amenability of the Projective Module Tensor Product

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ABSTRACT

Let S be an inverse semigroup with the set of idempotents E. In the current paper, we show that the projective module tensor product $\ell^1(S)\ \widehat{\otimes}_{l^1(E)}\ \ell^1(S)$ is $\ell^1(E)$-module amenable when S is amenable. This could be considered as the module version (for inverse semigroups) of a result of Johnson (1972) which asserts that for any (discrete) amenable locally compact group G (when $\ell^1(E)=\mathbb{C}$, the set of complex numbers), the projective tensor product $\ell^1(G)\ \widehat{\otimes}\ \ell^1(G) \cong \ell^1(G\times G)$ is amenable.

Keywords: Amenability, module amenability, module derivation, semigroup algebras.

INTRODUCTION

Let G be a discrete group. It is well known that the group algebra $\ell^1(G)$ is amenable if and only if G is amenable (1972). This fact fails for discrete semigroups. In fact, Duncan and Namioka (1988) proved that if the subsemigroup E of idempotent elements of inverse semigroup S is infinite, then the semigroup algebra $\ell^1(S)$ is not amenable. Amini (2004) introduced the concept of module amenability for a class of Banach algebras and showed that under some natural conditions for an inverse semigroup S with the set of idempotents E, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$ if and only if S is amenable. Now, for an amenable discrete group G, it follows from the celebrated Johnson's theorem (1972) that the projective tensor product $\ell^1(G)\widehat{\otimes}\ell^1(G)\cong \ell^1(G\times G)$ is amenable. This is not true for any discrete semigroup. In this paper, we prove that if S is an amenable inverse semigroup with the set of idempotents E, then $\ell^1(S)\widehat{\otimes}\ell^1(S)\cong \ell^1(S\times S)$ is module amenable as an $\ell^1(E)$-module. As a consequence, we prove that Banach $\ell^1(E)$-module $\ell^1(S)\ \widehat{\otimes}_{l^1(E)}\ \ell^1(S)$ is module amenable.

NOTATIONS AND PRELIMINARIES RESULTS

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A derivation from $\mathcal{A}$ into X is a bounded linear map $D: \mathcal{A} \longrightarrow X$ satisfying:

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in \mathcal{A}).$$

For each $x \in X$ the map $ad_x\,(a) = a.x - x.a$ for all $a \in \mathcal{A}$, is a derivation which is called an inner derivation. If X is a Banach $\mathcal{A}$-bimodule, so is X^*(the dual space of X). A Banach algebra $\mathcal{A}$ is called amenable if for any $\mathcal{A}$-bimodule X, every derivation $D: \mathcal{A} \longrightarrow X^*$ is inner.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions as follows:

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha), \ (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha.(a.x) = (\alpha.a).x, a.(\alpha.x) = (a.\alpha).x,$$
$$\alpha.(x.a) = (\alpha.x).a \quad (x \in X, a \in \mathcal{A}, \alpha \in \mathfrak{A}),$$

and similar for the right or two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module and $\alpha.x = x.\alpha$ for all $x \in X$ and $\alpha \in \mathfrak{A}$, then we say that X is a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D: \mathcal{A} \longrightarrow X$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b),$$
$$D(ab) = D(a).b + a.D(b),$$
$$D(\alpha.a) = \alpha.D(a), \qquad D(a.\alpha) = D(a).\alpha,$$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. If X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then each $x \in X$ define a module derivation as follows:

$$D_x\,(a) = a.x - x.a \quad (a \in \mathcal{A}),$$

and that is called *inner derivation.* A Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module module X, each module derivation $D: \mathcal{A} \longrightarrow X^*$ is inner; Amini (2004).
Let $\mathcal{A} \widehat{\otimes} \mathcal{A}$ be the projective tensor product of $\mathcal{A}$ and $\mathcal{A}$ which is a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule by the following actions:

$$\alpha.(a \otimes b) = (\alpha.a) \otimes b,\ \mathrm{c}.(a \otimes b) = (ca) \otimes b \quad (a, b, c \in \mathcal{A}, \alpha \in \mathfrak{A}),$$

and similar for the right actions. Then, the Rieffel's result (1978) shows that

$$\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A} \cong (\mathcal{A} \widehat{\otimes} \mathcal{A})/I,$$

where I is the closed linear span of

$$\{a.\alpha \otimes b - a \otimes \alpha.b : a, b \in \mathcal{A}, \alpha \in \mathfrak{A}\}.$$

Consider $\omega: \mathcal{A} \widehat{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$ defined by $\omega(a \otimes b) = ab$ and extend by linearity and continuity. Let also J be the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then I and J are both $\mathcal{A}$-submodules and $\mathfrak{A}$-submodules of $\mathcal{A} \widehat{\otimes} \mathcal{A}$ and $\mathcal{A}$, respectively. So $\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A}$ and $\mathcal{A}/J$ are both Banach $\mathcal{A}$-modules and $\mathfrak{A}$-modules. Specially, $\mathcal{A}/J$ is always an $\mathcal{A}$-$\mathfrak{A}$-module when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.

Define $\widetilde{\omega}: (\mathcal{A} \widehat{\otimes} \mathcal{A})/I \longrightarrow \mathcal{A}/J$ by $\widetilde{\omega}(a \otimes b + I) = ab + J$ and extend by linearity and continuity. Obviously, $\widetilde{\omega}$ and its dual conjugate $\widetilde{\omega}^{**}: (\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A})^{**} \cong (\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}/I^{\perp\perp} \longrightarrow \mathcal{A}^{**}/J^{\perp\perp}$ are $\mathcal{A}$-module homomorphisms and $\mathfrak{A}$-module homomorphisms.

The following result is similar to a classical case for module amenable Banach algebras which has been proved by Amini (2004).

Proposition 1. If $\mathcal{A}$ and B are Banach algebras and Banach $\mathfrak{A}$-modules with compatible actions, and there is a continuous Banach algebra homomorphism and module homomorphism from $\mathcal{A}$ onto a dense subset of B, and $\mathcal{A}$ is module amenable, then so is B.

Corollary 2. Let $\mathcal{A}$ be Banach $\mathfrak{A}$-module. Then module amenability of $\mathcal{A} \widehat{\otimes} \mathcal{A}$ implies module amenability $\mathcal{A}/J \widehat{\otimes} \mathcal{A}/J$.

Proof. The map

$$\varphi: \mathcal{A} \widehat{\otimes} \mathcal{A} \longrightarrow \mathcal{A}/J \widehat{\otimes} \mathcal{A}/J$$

defined by

$$\varphi(a \otimes b) = (a + J) \otimes (b + J) \quad (a, b \in \mathcal{A}),$$

is an epimorphism and $\mathfrak{A}$-module homomorphism. Now, we can apply Proposition 1. ■

The following definition is given by Amini (2004).

Definition 3. A bounded net $\{\tilde{\xi}_j\}$ in $\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A}$ is called a module approximate diagonal if $\tilde{\omega}(\tilde{\xi}_j)$ is a bounded approximate identity of $\mathcal{A}/J$ and

$$\lim_j \| \tilde{\xi}_j . a - a . \tilde{\xi}_j \| = 0 \ \ (a \in \mathcal{A}).$$

An element $\widetilde{M} \in (\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A})^{**}$ is called a module virtual diagonal if

$$\tilde{\omega}^{**}(\widetilde{M}). a = a + J^{\perp\perp}, \ \ \widetilde{M}. a = a. \widetilde{M} \quad (a \in \mathcal{A}).$$

Note that the ideal J in this paper is defined to be the closed ideal of $\mathcal{A}$ generated by elements of the form $(a.\alpha)b - a(\alpha.b)$, for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, whereas Amini *et al.* (2010), considered it as the closed ideal of $\mathcal{A}$ generated by elements of the form $\alpha.ab - ab.\alpha$. These two ideals are the same for the inverse semigroup algebra $\ell^1(S)$ with the corresponding actions of $\ell^1(E)$, but the definition Amini *et al.* (2010), has the advantage that J is also a Banach $\mathfrak{A}$-submodule of $\mathcal{A}$. However, Proposition 2.4 of Amini (2004), remain valid with this new definition of J when $\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module as follows:

Theorem 4. Let $\mathcal{A} \widehat{\otimes}_{\mathfrak{A}} \mathcal{A}$ be an commutative $\mathcal{A}$-$\mathfrak{A}$-module. Then the following are equivalent:
(i) $\mathcal{A}$ is module amenable and $\mathcal{A}/J$ has a bounded approximate identity.
(ii) $\mathcal{A}$ has a module approximate diagonal.
(iii) $\mathcal{A}$ has a module virtual diagonal.

TENSOR PRODUCT OF SEMIGROUP ALGEBRAS

In this section, we investigate the module amenability of $\ell^1(S)\,\widehat{\otimes}_{l^1(E)}\,\ell^1(S)$ as $\ell^1(E)$-module, where S is an inverse semigroup with the set of idempotents E. A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there is a unique element s^*such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e^2 = e^* = e$. The set of idempotents of S is denoted by E.
There is a natural order on E defined by:

$$e \leq f \Leftrightarrow ef = e \quad (e, f \in E).$$

The set E is a semilattice and Howie (1976) showed that it is also a commutative subsemigroup of S. In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module when $\ell^1(E)$ act on $\ell^1(S)$ by convolution from right and trivially from left, that is:

$$\delta_e.\delta_s = \delta_s\,,\ \delta_s.\delta_e = \delta_s * \delta_e = \delta_{se}\ (s \in S, e \in E).$$

By the above actions, the ideal J is the closed linear span of

$$\{\delta_{set} - \delta_{st}: s, t \in S, e \in E\}.$$

We consider an equivalence relation on S as follows:

$$s \approx t \Leftrightarrow \delta_s - \delta_t \in J \quad (s, t \in S).$$

Since E is a semilattice, for given $e, f \in E$, $ef \in E$ and $ef \leq e, f$. By using the argument in the paragraph before Theorem 2.4 of Amini *et al.* (2010), we can show that $S/\approx$ is a group. One should note that when S is a discrete group, then $S = S/\approx$. Now, consider the congruence relation $\sim$ on S where, $s \sim t$ if and only if there is an $e \in E$ such that $se = te$. It is proved by Howie (1976) that the quotient semigroup $G_S := S/\sim$ is then a maximal group homomorphic image of S. It is also proved that $S/\approx$ is isomorphic to G_S by Pourmahmood (2010). For two Banach algebras $\ell^1(S)$ and $\ell^1(G_S)$, Rezavand *et al.* (2009), showed that $\ell^1(S)/J \cong \ell^1(G_S)$. With the above observation $\ell^1(G_S)$ has an $\ell^1(E)$-module structure.

Henceforth, for each $s \in S$, the equivalence class of s in $G_S = S/\approx$ denotes by $[s]$. Bodaghi (2010) has proven that if S is amenable and E is an upward direct set, then $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module amenable. The upward directed condition for E is strong and in fact in the next theorem we showed that it is redundant. Consequently, the hypothesis on E being upward directed can be eliminated and $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module amenable when S is amenable. We are now going to prove the main result in this paper.

Theorem 5. Let S be an inverse semigroup with the set of idempotents E. Then the following statements are equivalent:
(i) $\ell^1(G_S)\widehat{\otimes}\ell^1(G_S) \cong \ell^1(G_S \times G_S)$ is module amenable.
(ii) $\ell^1(G_S)\ \widehat{\otimes}\ \ell^1(G_S)$ is amenable.
(iii) $\ell^1(S)\widehat{\otimes}\ell^1(S) \cong \ell^1(S \times S)$ is module amenable.

Proof. $(i) \Leftrightarrow (ii)$: Obviously, the left action $\ell^1(E)$ on $\ell^1(G_S)$ is trivial. Also it is shown in Lemma of Amini (2004) that right action is also trivial, that is:

$$\delta_{[s]}.\delta_e = \delta_{[se]} = \delta_{[s]} \quad (t \in S, e \in E).$$

This shows that $\ell^1(G_S)$ is a commutative Banach $\ell^1(G_S)$- $\ell^1(E)$-module and $\ell^1(G_S)\ \widehat{\otimes}_{l^1(E)}\ \ell^1(G_S) \cong\ \ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$. Thus every module approximate diagonal for Banach algebra $\ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$ is an approximate diagonal and vice versa. Therefore the result follows from Theorem 4 and Theorem 2.9.65 of Dales (2000).

$(iii) \Longrightarrow (i)$: In Corollary 2, put $\mathcal{A} = \ell^1(S)$, $\mathcal{A}/J = \ell^1(G_S)$ and $\mathfrak{A} = \ell^1(E)$.

$(i) \Longrightarrow (iii)$: Assume that X is a commutative Banach $\ell^1(S)\widehat{\otimes}\ell^1(S)$-$\ell^1(E)$-module with compatible actions. We consider the following module actions $\ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$ on X,

$$(\delta_{[s]} \otimes \delta_{[t]}).x = (\delta_s \otimes \delta_t).x$$
$$x.(\delta_{[s]} \otimes \delta_{[t]}) = x.(\delta_s \otimes \delta_t),$$

for all $t, s \in S, x \in X$. Indeed, $\delta_s - \delta_{se} \in J$ if and only if $\delta_{st} - \delta_{set} \in J$, for all $s, t \in S, e \in E$.

Now, for each $t, s \in S, x \in X$, and $e, f \in E$, we have

$$\begin{aligned}((\delta_s - \delta_{se}) \otimes (\delta_t - \delta_{tf})).x &= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\ &-(\delta_s \otimes \delta_{tf}).x + (\delta_{se} \otimes \delta_{tf}).x \\ &= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\ &-(\delta_s \otimes \delta_t).(x.\delta_f) + (\delta_{se} \otimes \delta_t).(x.\delta_f) \\ &= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\ &-((\delta_s \otimes \delta_t).x).\delta_f + ((\delta_{se} \otimes \delta_t).x).\delta_f \\ &= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\ &-(\delta_f.\delta_s \otimes \delta_t).x + (\delta_f.\delta_{se} \otimes \delta_t).x \\ &= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\ &-(\delta_s \otimes \delta_t).x + (\delta_{se} \otimes \delta_t).x = 0.\end{aligned}$$

Thus X becomes a commutative Banach $\ell^1(G_s)\widehat{\otimes}\ell^1(G_s)$-$\ell^1(E)$-module with compatible actions. Suppose that $D: \ell^1(S)\widehat{\otimes}\ell^1(S) \longrightarrow X^*$ is a module derivation. Define the map

$$\widetilde{D}: \ell^1(G_s)\widehat{\otimes}\ell^1(G_s) \longrightarrow X^*$$

via $\widetilde{D}(\delta_{[s]} \otimes \delta_{[t]}) := D(\delta_s \otimes \delta_t)$, for all $t, s \in S$, and extend by linearity. Since G_s is a discrete group, the group algebra $\ell^1(G_s)$ has an identity $\mathcal{E} = \mathfrak{e} + J$ ($\mathfrak{e} \in \ell^1(S)$). By definition of the map $\widetilde{D}$, we get

$$D(\delta_s \otimes \delta_{tu}) = D(\mathfrak{e}.\delta_s \otimes \delta_{tu}) \quad (s, t, u \in S).$$

Using the above equality we can show that $\widetilde{D}$ is well-defined. Due to module amenability of $\ell^1(G_s) \widehat{\otimes} \ell^1(G_s)$, the derivation D is inner. This completes the proof. ■

It is proved by Amini (2004) that if $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and trivially from left, then

$$\ell^1(S) \widehat{\otimes}_{l^1(E)} \ell^1(S) \cong \ell^1(S \times S)/I,$$

where I is the closed ideal of $\ell^1(S \times S)$ generated by the set of elements of the form $\delta_{(set,x)} - \delta_{(st,x)}$, where $s, t, x \in S, e \in E$.

Corollary 6. If S is an amenable inverse semigroup with the set of idempotents E, then $\ell^1(S)\,\widehat{\otimes}_{l^1(E)}\,\ell^1(S)$ is module amenable.

Proof. The semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable by Amini (2004), and so $\ell^1(G_s)$ is amenable by Amini *et al.* (2010). Thus $\ell^1(G_s)\,\widehat{\otimes}\,\ell^1(G_s)$ is amenable by Johnson's theorem (the projective tensor product of amenable Banach algebras is also amenable) . Now, the result follows from Proposition 1 and Theorem 5. ■

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